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All quadratic equations of the form

$$a+h(m+x)+bmx+g(m^2+x^2)+f(m^2x+x^2m)+cm^2x^2=0$$

(in which the first member is a symmetric function $f(\overset{\overset{2}{\smile}}{x}, \overset{\overset{2}{\smile}}{m})\equiv f(\overset{\overset{2}{\smile}}{m}, \overset{\overset{2}{\smile}}{x})$ *of its two arguments* x *and* m *of the second degree in each of them), whose parameters are related by the equation*

$$(b-g)g+ac-fh=0,$$

—and, apart from the relatively trivial equation

$$g(x^2-2mx+m^2)=0,$$

only those equations whose parameters are so related—are characterized by the property that between the two roots x_1, x_2 *which are functions of* m *the relation*

$$f(\overset{\overset{2}{\smile}}{x_1}, \overset{\overset{2}{\smile}}{x_2})\equiv 0$$

holds as an identity in m.

November 26, 1895.

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QUADRATURE OF THE CIRCLE.

By **WILLIAM E. HEAL, A. M.**, Member of the London Mathematical Society, and Treasurer of Grant County, Marion, Indiana.

The problem of the quadrature of the circle, or what amounts to the same thing, drawing a straight line equal in length to the circumference of a given circle, occupied the attention of mathematicians at a very early date. Long before the time of Archimedes, geometers had attacked the problem with but one result: failure. And for more than twenty centuries mathematicians have been struggling with the problem. Many claimed to have solved it, but their analysis has been, in every case, found to be fatally defective. After centuries of attempt and failure mathematicians began to suspect that the problem might not admit of solution. James Gregory was the first to attempt a proof of the impossibility of the quadrature of the circle. In the opinion of Montucla he succeeded; but later mathematicians have not so decided. Not a score of years have passed since a rigid proof was given that the solution of the problem is really impossible under the conditions usually understood: that is, by the use of the rule and compass only.

It is well known from the geometry that the ratio of the circumference to the diameter of a circle is constant. This constant ratio is usually denoted by the Greek letter π, and it follows at once that if π is a number commensurable with unity that it can be constructed geometrically, and the problem is solved. Lambert, in a memoir presented to the Berlin academy in 1761, was the first to prove that π is incommensurable. Other proofs of this result have been given, especially by Hermite in Crelle's Journal, Vol. 76, which demonstration is reproduced in the Traite de Geometrie of Rouche and Comberousse, 4th edition. But this result, however interesting in itself, does not prove the impossibility of a geometrical construction of π. For example, the square root of 2 (or any non-quadrate number) is incommensurable but is easily constructed geometrically. The first real advance towards the solution of the problem was made by Hermite in 1873. Hermite succeeded in proving that the number e, the base of the Naperian system of logarithms is not only incommensurable but that it can not be a root of a rational algebraic equation of any degree whatever. Such a number is called transcendent. If the number π could be proved to be transcendent the vexed question of the quadrature of the circle would be settled once for all. For this problem requires to derive the number π by a finite number of elementary geometrical constructions. As two straight lines, or a straight line and a circle, or two circles, have not more than two intersections, these processes, or any finite combination of them, can be expressed algebraically in a comparatively simple form ; so that the solution of the problem of the quadrature of the circle would mean that π can be expressed as the root of an algebraic equation solvable by square roots. Hermite did not succeed in proving that π is a transcendent number, but in 1882 Lindemann extended Hermite's proof to include the number π as well as e among the transcendent numbers. Hermite and Lindemann's methods are complicated and obscure and many mathematicians attempted to simplify them. But not until very recently were these attempts rewarded with any degree of success. In January, 1893, Hilbert published a proof of the transcendency of e and π that reduces the problem to such simple terms as to be understood by mathematicians having only a moderate understanding of the principles of the calculus. Hilbert's proof depends upon certain properties of the definite integral

$$\int_0^\infty z^\rho \Big[(z-1)(z-2)(z-3)\ldots\ldots(z-n)\Big]^{\rho+1} e^{-z}dz,$$

suggested by the investigations of Hermite.

Immediately after the publication of Hilbert's proof, Hurwitz published a proof for the transcendency of e based on still more elementary principles. And finally, in May, 1893, Gordan published a proof of the transcendency of e and π in which only the known development of e^x in powers of x is made use of. This last proof is so simple that it should be introduced into university teaching everywhere. The numbers e and π are very intimately related, and before proceeding to Gordan's proof of the transcendency of these two fundamental numbers I wish to give here a well known proof that e is incommensurable. We have

$$e=1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\ldots\ldots\frac{1}{\underline{|r}}+\frac{1}{\underline{|r+1}}\ldots\ldots$$

Assume, now, that e is a rational number $\frac{a}{r}$, where a and r are integers, and the fraction $\frac{a}{r}$ is in its lowest terms. Multiply this equation by $\underline{|r}$ and we see that all the terms preceding the term $\frac{1}{\underline{|r+1}}$ are integers. The series

$$\frac{1}{r+1}+\frac{1}{(r+1)(r+2)}+\frac{1}{(r+1)(r+2)(r+3)}\ldots\ldots \text{ is less than}$$

$$\frac{1}{r+1}+\frac{1}{(r+1)^2}+\frac{1}{(r+1)^3}\ldots\ldots$$

That is less than $\frac{1}{r}$. Thus we have an integer equal to a proper fraction which is impossible. I will now give Gordan's proof of the transcendency of e and π. The proof for e will be seen to be an extension of the above well-known proof of the irrationality of e and apparently should have been discovered long ago.

The function e^x is defined by the series

$$e^x=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}\ldots\ldots$$

This, if we introduce the symbolic notation

$$\underline{|r}=h^r$$

and multiply by this quantity and any whole number c_r, passes into the form

$$c_r h^r e^x=c_r(x+h)^r+c_r x^r u_r \tag{1}$$

in which

$$u_r=\frac{x}{r+1}+\frac{x^2}{(r+1)(r+2)}\ldots\ldots$$

If

$$\mu=\text{mod.}x$$

we have

$$\text{mod.}u_r<e^{\mu};$$

and if we put

$$u_r=q_r e^{\mu},$$

$$\text{mod.}q_r<1.$$

From (1) it follows:

$$c_r h^r e^x = c_r (x+h)^r + c_r x^r q_r e^{\mu}.$$

And $$e^x \sum_{r=0}^{r=8} c_r h^r = \sum_{r=0}^{r=8} c_r (x+h)^r + e^{\mu} \sum_{r=0}^{r=8} c_r q_r x^r.$$

And if we put $$\sum_{r=0}^{r=8} c_r x^r = \phi(x), \ \sum_{r=0}^{r=8} c_r q_r x^r = \psi(x);$$

$$(2) \qquad e^x \phi(h) = \phi(x+h) + e^{\mu} \psi(x).$$

If, now, there is an equation with integral coefficients, satisfied by the number e:

$$\sum_{k=o}^{k=n} c_k e^k = 0,$$

then from (2) we have

$$(3) \qquad 0 = \sum_{k=o}^{k=n} c_k \phi(k+h) + \sum_{k=o}^{k=n} c_k \psi(k) e^k.$$

If we choose for ϕ the function

$$\phi(x) = \frac{x^{p-1}}{\underline{|p-1}} [(x-1)(x-2 \ldots\ldots (x-n)]^p$$

and for ρ a prime number greater than the numbers n and c_0, then will $\phi(k+h)$ in formula (3) become whole numbers.

$$\phi(h+1), \ \phi(h+2) \ldots\ldots \phi(h+n)$$

have the factor p, but

$$c_0 \phi(h)$$

has not. If we let p increase, then ϕ and ψ become as small as we please, and formula (3) is impossible, and the number e transcendent.

If $i\pi$ is a root of an equation with integral coefficients:

$$(4) \qquad c(x-w_1)(x-w_2) \ldots\ldots (x-w_{\rho}) = 0,$$

then we have the formula

$$(5) \qquad (1+e^{w_1})(1+e^{w_2}) \ldots\ldots (1+e^{w_{\rho}}) = 0.$$

If $c-1$ vanishing quantities are found among the sums

$$w_i\,;\ w_i+w_k\,;\ w_i+w_k+w_\lambda\,\ldots\ldots$$

and we designate those remaining by

$$a_1,\ a_2,\ a_3\ldots\ldots a_n$$

and their moduli by

$$a_1,\ a_2,\ a_3\ldots\ldots a_n,$$

formula (5) becomes

$$0=c+\sum_{k=1}^{k=n} e^{a_k}. \tag{6}$$

The symmetric functions of cw_k, as well as those of ca_k, are whole numbers. By formula (2) we have

$$0=c\phi(h)+\sum_{k=1}^{k=n}\phi(a_k+h)+\sum_{k=1}^{k=n}e^{a_k}\psi(a_k). \tag{7}$$

Let $$\phi(x)=\frac{(cx)^{p-1}}{\underline{|p-1}}\,c^{np}[(x-a_1)(x-a_2)\ldots\ldots(x-a_n)]^p,$$

and let p be a prime number greater than the numbers

$$c\,;\ n\,;\ c\,;\ c^n a_1 a_2\ldots\ldots a_n.$$

The quantities $\phi(h)$ and $\sum_{k=1}^{k=n}\phi(a_k+h)$ are whole numbers :

$$\sum_{k=1}^{k=n}\phi(a_k+h)$$

contains the factor p, but $c\phi(h)$ does not.

If p increases the moduli of ϕ and ψ become as small as we please.

Formula (7) is impossible, and therefore π is a transcendent number.